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The stability of thermally stratified plane Poiseuille flow

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In studying the stability of a thermally stratified fluid in the presence of a viscous shear flow, we have a situation in which there is an important interaction between the mechanism of instability due to the stratification and the Tollmien–Schlichting mechanism due to the shear. A complete analysis has been carried out for the Bénard problem in the presence of a plane Poiseuille flow and it is shown that, although Squire's transformation can be used to reduce the three-dimensional problem to an equivalent two-dimensional one, a theorem of Squire's type does not follow unless the Richardson number exceeds a certain small negative value. This conclusion follows from the fact that, when the stratification is unstable and the Prandtl number is unity, the equivalent two-dimensional problem becomes identical mathematically to the stability problem for spiral flow between rotating cylinders and, from the known results for the spiral flow problem, Squire's transformation can then be used to obtain the complete three-dimensional stability boundary. For the case of stable stratification, however, Squire's theorem is valid and the instability is of the usual Tollmien–Schlichting type. Additional calculations have been made for this case which show that the flow is completely stabilized when the Richardson number exceeds a certain positive value.

1. Introduction

The effects of stratification on the stability of inviscid, parallel shear flows have been extensively studied in recent years (see, for example, Miles 1961 or Drazin & Howard 1966) and the theory of such flows would now appear to be reasonably complete. Attempts to extend this theory to include the effects of viscosity and thermal conductivity (assuming that the stratification is of thermal origin) have thus far been only partially successful. For example, the early work by Schlichting (1935) on the stability of stratified boundary layer flows included only the effects of viscosity and, as a result, the linearized disturbance equation that he used was singular. More recent work by Kuo (1963), Deardorff (1965) and Gallagher & Mercer (1965) on the stability of thermally stratified plane Couette flow avoids this particular difficulty but it fails, due to the intrinsic stability of plane Couette flow, to exhibit one important feature of the general problem, namely, the interaction between the Tollmien–Schlichting mechanism of instability associated with the shear flow and the thermal instability associated with an adverse temperature gradient. In the present paper, therefore, we have considered a problem with two well-defined stability limits (in the absence of shear it reduces to the Bénard problem and in the absence of stratification it reduces to the problem of plane Poiseuille flow) which then permits a complete discussion not only of the interaction between the two mechanisms of instability but also of the important relationship between two- and three-dimensional disturbances which appears to be typical for this class of problems.

Consider then the stability of a viscous, heat conducting fluid confined between the planes $y = \pm\frac{1}{2}d$. We suppose that the mean velocity is in the x-direction and that there is a mean temperature gradient maintained in the y-direction. The two main parameters of the problem are then the Reynolds number R associated with the basic flow and the Rayleigh number Ra associated with the mean temperature gradient. When $R = 0$ we know that instability sets in with respect to arbitrary three-dimensional disturbances at a critical Rayleigh number of 1708, whereas when $Ra = 0$ the instability sets in with respect to two-dimensional Tollmien–Schlichting waves at a critical Reynolds number of about 5400. For some purposes it is also convenient to introduce the Richardson number $Ri = -Ra/64R^2P$, where P is the Prandtl number, which is independent of both viscosity and thermal conductivity. Positive values of the Richardson number then correspond to stable stratification and conversely.

In the general treatment of this problem it is clearly essential to consider three-dimensional disturbances. As Koppel (1964) has shown, Squire's (1933) transformation can be used to reduce this three-dimensional problem to an equivalent two-dimensional one. From the solution of this equivalent two-dimensional problem it is then possible to deduce the required three-dimensional stability boundary and it is found that Squire's theorem holds (i.e. that two-dimensional disturbances are more unstable than three-dimensional ones) if and only if $Ri > Ri_*$, where Ri_* is small and negative. This conclusion, however, appears to be at variance with Koppel's generalization of Squire's theorem on the basis of which he considered only two-dimensional disturbances.

There is thus an important difference in the character of the problem depending upon whether the stratification is stable or unstable. When $Ri < 0$ and $P = 1$ there is an exact mathematical analogy between the governing equations of the present problem and the problem of the stability of spiral flow between rotating cylinders (Hughes & Reid 1968). By exploiting this analogy together with Squire's transformation it is then very easy to obtain the complete stability boundary in the (R, Ra)-plane from the known results for the spiral flow problem. When $Ri > 0$, however, further calculations are required and these have also been made for $P = 1$. The method of calculation is based on the asymptotic methods of approximation previously developed for the spiral flow problem in which we let $R \to \infty$ for fixed values of Ri. These calculations show that when $Ri > 0{\cdot}0554$ the flow is completely stable. This conclusion is perhaps not unexpected since plane Poiseuille flow is only weakly unstable and its instability is viscous in origin.

2. The governing equations

The linearized disturbance equations which govern the stability of a thermally stratified parallel shear flow have been derived previously by Koppel (1964). In deriving these equations the usual Boussinesq approximation was also made. For the present purposes it is convenient to non-dimensionalize the equations in terms of a characteristic length $L_* = \frac{1}{2}d$, a characteristic velocity U_* equal to the maximum velocity of the basic flow, and a characteristic temperature $T_* = \frac{1}{2}\Delta T$, where ΔT is the imposed temperature difference between the two bounding planes. The Reynolds, Rayleigh and Richardson numbers can then be defined in the usual way by

$$R = \frac{U_* d}{2\nu}, \quad Ra = -\frac{\gamma g d^3 \Delta T}{\kappa \nu} \quad \text{and} \quad Ri = \frac{\gamma g d \Delta T}{16 U_*^2}, \tag{2.1}$$

where γ is the coefficient of thermal expansion.

If we now assume that all of the disturbances have a dependence on x, z and t of the form $\exp\{i(\alpha x + \beta z) - i\alpha c t\}$ then we obtain the equations

$$\left.\begin{aligned} \{D^2 - (\alpha^2+\beta^2) - i\alpha R(U-c)\}u &= i\alpha R p + R U' v, \\ \{D^2 - (\alpha^2+\beta^2) - i\alpha R(U-c)\}v &= R D p - 4 Ri R \theta, \\ \{D^2 - (\alpha^2+\beta^2) - i\alpha R(U-c)\}w &= i\beta R p, \\ i(\alpha u + \beta w) + Dv &= 0 \\ \text{and} \quad \{D^2 - (\alpha^2+\beta^2) - i\alpha R P(U-c)\}\theta &= R P \Theta' v, \end{aligned}\right\} \tag{2.2}$$

where $U(y)$ and $\Theta(y)$ are the mean velocity and temperature distributions respectively and $D = d/dy$. These equations, together with the boundary conditions

$$u = v = w = \theta = 0 \quad \text{at} \quad y = \pm 1, \tag{2.3}$$

define the three-dimensional problem.

We now wish to show that this three-dimensional problem can be reduced, by means of Squire's transformation, to an equivalent two-dimensional problem. For this purpose we let

$$\left.\begin{aligned} \tilde{\alpha}\tilde{u} = \alpha u + \beta w, \quad \tilde{v} = v, \quad \tilde{p} = (\tilde{\alpha}/\alpha) p, \quad \tilde{\theta} = (\alpha/\tilde{\alpha})\theta, \\ \tilde{\alpha} = (\alpha^2+\beta^2)^{\frac{1}{2}}, \quad \tilde{c} = c, \quad \tilde{P} = P, \quad \tilde{R} = (\alpha/\tilde{\alpha}) R, \quad \tilde{Ri} = (\tilde{\alpha}/\alpha)^2 Ri. \end{aligned}\right\} \tag{2.4}$$

The equations (2.2) are thereby reduced to the form

$$\left.\begin{aligned} \{D^2 - \tilde{\alpha}^2 - i\tilde{\alpha}\tilde{R}(U-\tilde{c})\}\tilde{u} &= i\tilde{\alpha}\tilde{R}\tilde{p} + \tilde{R} U' \tilde{v}, \\ \{D^2 - \tilde{\alpha}^2 - i\tilde{\alpha}\tilde{R}(U-\tilde{c})\}\tilde{v} &= \tilde{R} D \tilde{p} - 4 \tilde{Ri} \tilde{R} \tilde{\theta}, \\ i\tilde{\alpha}\tilde{u} + D\tilde{v} &= 0 \\ \text{and} \quad \{D^2 - \tilde{\alpha}^2 - i\tilde{\alpha}\tilde{R}\tilde{P}(U-\tilde{c})\}\tilde{\theta} &= \tilde{R}\tilde{P}\Theta'\tilde{v}. \end{aligned}\right\} \tag{2.5}$$

These equations, together with the boundary conditions

$$\tilde{u} = \tilde{v} = \tilde{\theta} = 0 \quad \text{at} \quad y = \pm 1, \tag{2.6}$$

obviously have the same mathematical structure as equations (2.2) and (2.3) with $\beta = w = 0$ and they thus define the equivalent two-dimensional problem. Koppel's generalization of Squire's theorem immediately follows from this result and states that 'the three-dimensional problem is equivalent to a two-dimensional problem at a smaller Reynolds number and a larger Richardson number'. Since the Richardson number is not invariant with respect to Squire's transformation, however, it does *not* follow that we can simply consider two-dimensional disturbances from the outset, i.e. simply set $\beta = w = 0$ in (2.2) and (2.3). Instead we must consider the equivalent two-dimensional problem defined by (2.5) and (2.6); from the solution of this problem we can then derive, by means of the transformation (2.4), the required solution of the three-dimensional problem.

We now introduce a stream function $\phi(y)\exp\{i\tilde{\alpha}(\tilde{x}-\tilde{c}t)\}$ in the usual way, where $\tilde{\alpha}\tilde{x} = \alpha x + \beta z$, so that $\tilde{u} = \phi'$ and $\tilde{v} = -i\tilde{\alpha}\phi$. The equations which govern the equivalent two-dimensional problem can then be written in the form

$$L_4\phi = 4\tilde{R}i\tilde{\theta} \quad \text{and} \quad L_2\tilde{\theta} = -\Theta'\phi, \tag{2.7}$$

where
$$\left.\begin{aligned} L_2 &= (i\tilde{\alpha}\tilde{R}\tilde{P})^{-1}(D^2-\tilde{\alpha}^2)-(U-\tilde{c}) \\ L_4 &= (i\tilde{\alpha}\tilde{R})^{-1}(D^2-\tilde{\alpha}^2)^2-(U-\tilde{c})(D^2-\tilde{\alpha}^2)+U''.\end{aligned}\right\} \tag{2.8}$$
and

On eliminating $\tilde{\theta}$ between these equations we have

$$L_2L_4\phi = -4\tilde{R}i\Theta'\phi, \tag{2.9}$$

together with the boundary conditions

$$\phi = \phi' = L_4\phi = 0 \quad \text{at} \quad y = \pm 1. \tag{2.10}$$

In the present problem the mean velocity and temperature distributions have the simple forms

$$U(y) = 1-y^2 \quad \text{and} \quad \Theta(y) = y. \tag{2.11}$$

But, just as in the usual stability theory for nearly parallel shear flows, the linearized disturbance equations derived here may be applicable to more general classes of velocity and temperature distributions.

3. Results for unstable stratification

When $Ri < 0$ and $P = 1$, the equivalent two-dimensional problem defined by equations (2.9) and (2.10) becomes mathematically identical with the equations that govern the stability of spiral flow between rotating cylinders provided we identify the Rayleigh number in the present problem with the Taylor number in the spiral flow problem. The relationship between these two problems is thus a simple generalization of the well-known analogy between the usual Bénard and Taylor problems. The results obtained recently by Hughes & Reid (1968) for the spiral flow problem provide the required solution of the equivalent two-dimensional problem from which, by means of Squire's transformation, we can now obtain the solution of the three-dimensional problem.

The precise relationship between the two- and three-dimensional problems follows immediately from the fact that *the Rayleigh number remains invariant under Squire's transformation*. Thus, if we let $\alpha/\tilde{\alpha} = \cos\lambda$ say, so that $\tilde{R} = R\cos\lambda$ and $\tilde{R}a = Ra$, then we can easily obtain the family of stability boundaries shown in figure 1. The curve labelled $\lambda = 0°$ is the stability boundary of the equivalent two-dimensional problem obtained from the solution of the spiral flow problem and the other curves follow from it by a simple translation. From this construction it is clear that we have stability only for that part of the $(R, \sqrt{Ra})$-plane which is interior to the envelope of these curves.

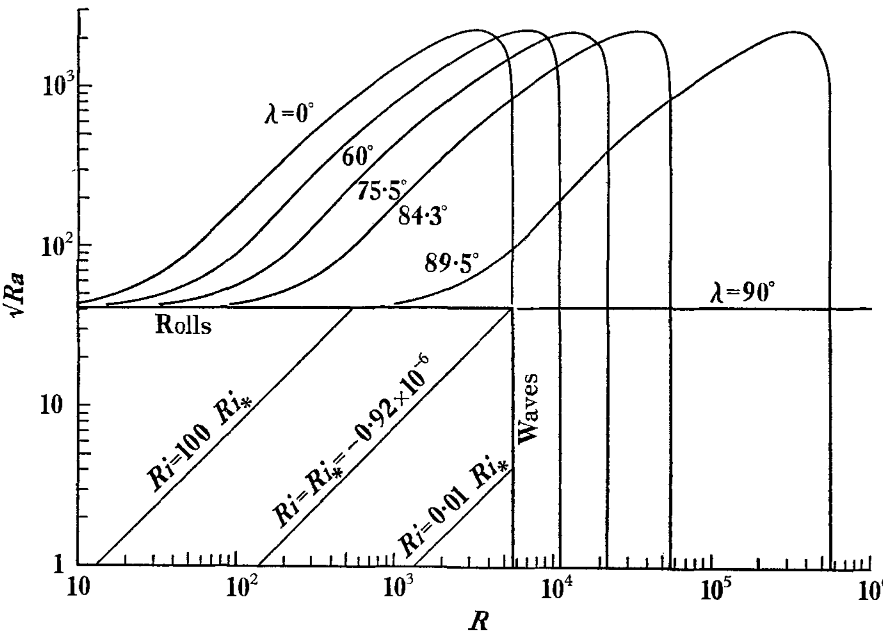

FIGURE 1. The construction of the three-dimensional stability boundary from the solution of the equivalent two-dimensional problem for the case of unstable stratification.

Thus, the required three-dimensional stability boundary consists of two parts. On one part of the boundary ($\lambda = 90°$ and $Ra = 1708$) the instability is purely thermal in origin and leads to steady convection in the form of longitudinal rolls whose axes are in the direction of the mean flow. On the other part of the boundary ($\lambda = 0°$ and $R = 5400$), however, the instability leads to the usual two-dimensional Tollmien–Schlichting waves.

Since the Richardson number varies monotonically along the stability boundary, there exists a unique value $Ri_* = -1708/64(5400)^2 = -0{\cdot}92 \times 10^{-6}$ which marks the abrupt transition from one type of instability to the other. For $Ri < Ri_*$, Squire's theorem is not valid and it is necessary to determine the stability boundary as outlined above; for $Ri > Ri_*$, however, Squire's theorem is valid and we need only consider two-dimensional disturbances. The smallness of Ri_* serves to emphasize the dominant role of thermal instability for negative values of the Richardson number.

4. Results for stable stratification

When the Richardson number is positive, the effect of the stratification is purely stabilizing. Although the governing equations for the equivalent two-dimensional problem with $P = 1$ retain the same form as the equations governing the spiral flow problem, the sign of the term on the right-hand side of (2.9) is now negative and additional calculations are therefore required. Fortunately, however, these calculations are substantially simpler than in the case of the spiral flow problem since it was found to be unnecessary to use either of the viscous corrections to the singular inviscid solutions or the composite viscous solutions of the Tollmien type. Furthermore, since Squire's theorem is now valid, we need only consider two-dimensional disturbances and we can therefore drop the tildes throughout the subsequent discussion. The governing equation then becomes

$$L_2 L_4 \phi = -4Ri\phi \tag{4.1}$$

and for an even solution we impose the boundary conditions

$$\phi = \phi' = L_4\phi = 0 \quad \text{at} \quad y = -1 \quad \text{and} \quad \phi' = \phi''' = \phi^v = 0 \quad \text{at} \quad y = 0. \tag{4.2}$$

The general asymptotic theory for an equation of this type has recently been discussed in detail by Hughes & Reid (1968) and we shall therefore describe briefly only those parts of the theory that are essential for the present purposes or which require further modification. The approximations to the solutions of (4.1) that will be used here are based on letting $R \to \infty$ for fixed values of Ri and, as in the usual stability theory for parallel shear flows, they involve solutions of the so-called viscous and non-viscous type. In this discussion we shall defer temporarily the imposing of the condition that $P = 1$.

Consider first a formal expansion of the solution in inverse powers of $i\alpha R$ of the form

$$\phi(y) = \phi^{(0)}(y) + (i\alpha R)^{-1}\phi^{(1)}(y) + \ldots, \tag{4.3}$$

where $\phi^{(0)}(y)$ satisfies the inviscid equation

$$(U-c)^2(D^2-\alpha^2)\phi - (U-c)U''\phi + 4Ri\phi = 0. \tag{4.4}$$

This equation is independent of the Prandtl number and is, of course, familiar from the inviscid stability theory for parallel shear flows in a stratified fluid (see, for example, Drazin & Howard 1966). In the present context, however, we are primarily interested in the extent to which the solutions of this equation provide approximations to two solutions of equation (4.1). These solutions can conveniently be written in the form

$$\phi_1(y) = (y-y_c)^{p_1}P_1(y-y_c) \quad \text{and} \quad \phi_2(y) = (y-y_c)^{p_2}P_2(y-y_c), \tag{4.5}$$

where p_1 and p_2 are the roots of the indicial equation

$$p(p-1) + 4Ri/U_c'^2 = 0 \tag{4.6}$$

(with $p_1 > p_2$) and $P_1(y-y_c)$ and $P_2(y-y_c)$ are power series in $y-y_c$ with leading terms of unity. Both of these solutions have algebraic branch points at the critical point y_c where $U-c = 0$, and therefore neither of them can provide valid approximations in a full complex neighbourhood of y_c. By considering the viscous approximations to ϕ_1 and ϕ_2, however, Hughes & Reid (1968) have shown that they do provide valid asymptotic approximations in the usual sector

$$-\tfrac{7}{6}\pi < \arg(y-y_c) < \tfrac{1}{6}\pi$$

of the complex y-plane. Furthermore, since $0 < p_2 < \frac{1}{2} < p_1 < 1$ in the present calculations, the singularities in ϕ_1 and ϕ_2 are sufficiently mild that it is unnecessary to include either of the viscous corrections in the characteristic equation.

Consider next the approximations of viscous type. Since the values of c in the present problem never exceed the values found for plane Poiseuille flow, it is sufficient to consider only the so-called local turning-point approximations, thereby avoiding the further complications involved in the use of composite approximations of the Tollmien type. For this purpose we first make the transformation

$$\phi(y) = \chi(\xi) \tag{4.7}$$

where $\xi = (y-y_c)/\epsilon$ and $\epsilon = (i\alpha R U_c')^{-\frac{1}{3}}$,

and then expand the solution in powers of ϵ in the form

$$\chi(\xi, \epsilon) = \chi^{(0)}(\xi) + \epsilon\chi^{(1)}(\xi) + \dots. \tag{4.8}$$

The first approximation $\chi^{(0)}(\xi)$ then satisfies the equation

$$(P^{-1}D^2 - \xi)(D^2 - \xi)D^2\chi = -(4Ri/U_c'^2)\chi, \tag{4.9}$$

where D now stands for $d/d\xi$. Koppel (1964) has obtained integral representations for the solutions of this equation, but when $P \neq 1$ they are of a complicated form with kernels that involve Whittaker functions. When $P = 1$, however, it can easily be shown that all of the solutions of equation (4.9) can be obtained from the two third-order equations

$$\chi''' - \xi\chi' + p_i\chi = 0, \tag{4.10}$$

where p_i $(i = 1, 2)$ are the roots of the indicial equation (4.6). The solutions of equation (4.10) that have strictly neutral asymptotic expansions in the sector $-\frac{2}{3}\pi < \arg\xi < 0$ provide the first viscous corrections to the singular inviscid solutions and thus determine the sector of validity of ϕ_1 and ϕ_2. As was mentioned above, however, the uncorrected forms of ϕ_1 and ϕ_2 are completely adequate for the present purposes. Since we expect that viscous effects will be negligible in the central part of the channel, the only solutions of equation (4.10) that need be considered further are

$$\chi_3(\xi) = A_1(\xi, p_1) \quad \text{and} \quad \chi_5(\xi) = A_1(\xi, p_2), \tag{4.11}$$

where $A_1(\xi, p_i)$ are the solutions of (4.10) that are subdominant in the sector $|\arg\xi| < \frac{1}{3}\pi$. This last condition serves to define the solutions $A_1(\xi, p_i)$ uniquely to within a multiplicative constant.

To obtain an approximation to the characteristic equation it is convenient first to let

$$\Phi = A\phi_1 + \phi_2 \tag{4.12}$$

be the solution of the inviscid equation that satisfies the boundary condition $\Phi'(0) = 0$. Since $U(y)$ is an even function of y, Φ automatically satisfies the other two boundary conditions at $y = 0$. We then consider the approximation

$$\phi = \Phi + C_3\chi_3 + C_5\chi_5, \tag{4.13}$$

which satisfies the boundary conditions at $y = 0$ to within an exponentially small error, and the satisfaction of the three boundary conditions at $y = -1$ then leads to the required characteristic equation. The third of these boundary conditions, however, can be considerably simplified; for, consistent with the approximations already made, we have

$$L_4\Phi \to \frac{4Ri}{U-c}\Phi \quad \text{and} \quad L_4\chi \to \epsilon^{-1}U_c'(1-p)\chi'. \tag{4.14}$$

In this way we obtain the characterstic equation in the relatively simple form

$$\Delta(\alpha, c, z; Ri) \equiv \frac{1}{1+y_c}(p_1 - p_2) + \frac{\Phi'(-1)}{\Phi(-1)}\{p_1 F(z, p_1) - p_2 F(z, p_2)\} + \frac{U_c'}{c} p_1 p_2 \{F(z, p_1) - F(z, p_2)\} = 0, \tag{4.15}$$

where $z = (\alpha R U_c')^{\frac{1}{3}}(1+y_c)$ as usual and $F(z,p)$ is the generalized Tietjens function

$$F(z,p) = \frac{A_1(\xi, p)}{\xi A_1'(\xi, p)} \quad \text{with} \quad \xi = z e^{-\frac{5}{6}\pi i} \quad \text{and} \quad p \text{ real}. \tag{4.16}$$

For fixed values of Ri, the zeros of Δ define a curve of neutral stability in the (α, R)-plane. Numerical methods for the finding of these zeros are now well-known (see, for example, Mack 1965; Isaacson & Keller 1966; or Hughes & Reid 1968) and need not be repeated here. The calculations can be substantially simplified, however, if we let $\eta = 4Ri/U_c'^2$ and consider the values of η rather than Ri to be fixed. The parameter η can be thought of as a Richardson number based on the velocity gradient at the critical point rather than at the boundary and, since the values of c are always less than about 0·27, the difference between these two parameters can never become large (see table 1). The curves of neutral stability obtained in this way are shown in figure 2 and the corresponding behaviour of the wave-speed c is shown in figure 3. One interesting feature of these results is that as $R \to \infty$ for a fixed positive value of η, $\alpha \to \alpha_s > 0$ and $c \to 0$ along both branches of the curve. The approach to this limit will be examined in more detail below, but it should be observed here that this behaviour is consistent with the fact that the flow is stable in the inviscid limit; for finite values of the Reynolds number, the instability is entirely viscous in origin. For $\eta = 0$, there is a 'kink' in the neutral curve as shown in figure 2 which requires some comment. Computationally, it is due to the 'loop' in the ordinary Tietjens function; more basically, however, it would appear to be due to the lack of uniformity in the usual asymptotic approximations. Such kinks do not occur in the other curves shown in figure 2 simply because along them z never becomes as large as 6 (see table 2 below). As η (or Ri) approaches the value 0·0554, we approach the condition for complete stability as shown in figure 4, and for $\eta > 0·0554$ we would conclude that stratified plane Poiseuille flow is completely stable.

The limiting solution as $R \to \infty$ along the curves of neutral stability

As $R \to \infty$ along a curve of neutral stability, we find that $\alpha \to \alpha_s$ and $c \to 0$ along both branches but that z approaches different finite limits, $z_s^{\pm}$ say, along the upper and lower branches respectively. This situation would appear to be somewhat unusual and we shall therefore attempt a partial explanation of it. According to the results obtained by Miles (1961) in his study of the inviscid stability theory for stratified parallel flows, the limiting form of the inviscid solution, Φ_s say, must be a multiple of *either* ϕ_1 *or* ϕ_2. From the present numerical work it would appear that $A \to \infty$ in equation (4.12) as $c \to 0$ and hence that Φ_s is a multiple of ϕ_1. The limiting value of the wave-number α_s must be determined therefore as the 'eigenvalue' of the equation

$$U^2(D^2-\alpha_s^2)\,\phi_1 - UU''\phi_1 + 4Ri\phi_1 = 0, \tag{4.17}$$

together with the boundary condition $\phi_1'(0) = 0$.

η	p_1	p_2	Ri	z	α	c	$R_{cr}^{\frac{1}{3}} \times 10^{-2}$
0·000	1·000	0·000	0·0000	3·043	1·022	0·2672	0·1754
0·005	0·995	0·005	0·0038	3·08	1·005	0·2482	0·1925
0·010	0·990	0·010	0·0077	3·12	0·988	0·2285	0·2134
0·015	0·985	0·015	0·0119	3·15	0·964	0·2075	0·2396
0·020	0·980	0·020	0·0163	3·19	0·943	0·1861	0·2732
0·025	0·975	0·025	0·0210	3·23	0·918	0·1635	0·3182
0·030	0·969	0·031	0·0258	3·27	0·890	0·1399	0·3811
0·035	0·964	0·036	0·0310	3·31	0·859	0·1152	0·4753
0·040	0·959	0·041	0·0364	3·35	0·822	0·0891	0·6319
0·045	0·953	0·047	0·0422	3·39	0·781	0·0617	0·9415
0·050	0·947	0·053	0·0489	3·43	0·732	0·0329	1·833
0·052	0·945	0·055	0·0509	3·45	0·708	0·0209	2·931
0·054	0·943	0·057	0·0535	3·47	0·682	0·0090	6·963
0·0554	0·941	0·059	0·0554	3·485	0·656	0·0000	∞

TABLE 1. The values of the minimum critical Reynolds number and related parameters.

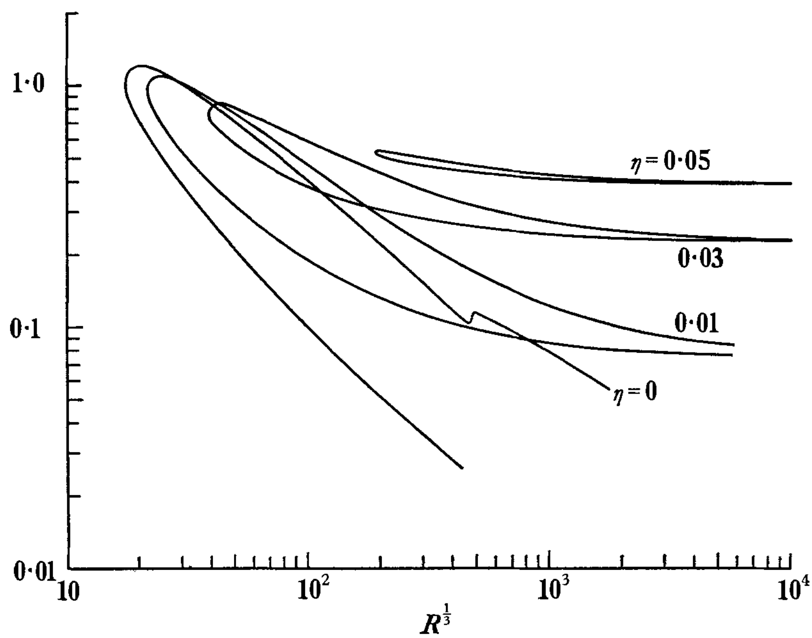

FIGURE 2. The curves of neutral stability for the case of stable stratification.

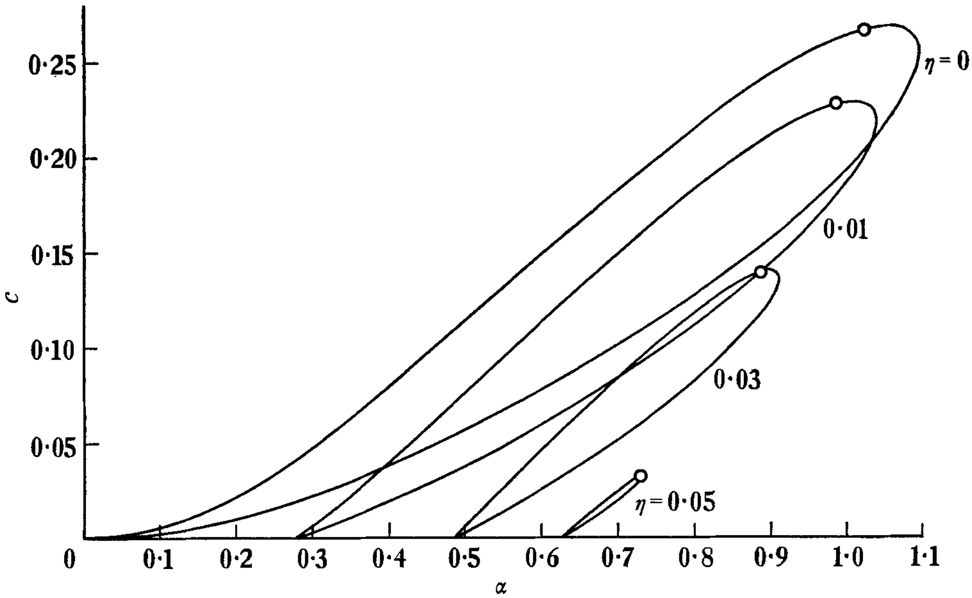

FIGURE 3. The relationship between the wave-number α and the wave-speed c along the neutral curves of figure 2. The circled points correspond to the minimum critical Reynolds number.

To obtain the limiting values of z_s, however, it is necessary to obtain the limiting form of the characteristic equation (4.15) as $R \to \infty$. For this purpose it is necessary to have a more precise estimate of the dependence of A on c as $c \to 0$. On closer examination of the numerical results it appears that

$$A \to a_s^{\pm} c^{-(p_1-p_2)} \quad \text{as} \quad c \to 0. \tag{4.18}$$

It then follows from (4.5) and (4.12) that

$$\left.\begin{aligned} \Phi(-1) &\to c^{p_2}(a_s^{\pm} 2^{-p_1} e^{-p_1 \pi i} + 2^{-p_2} e^{-p_2 \pi i}) \\ \text{and} \qquad (1+y_c)\Phi'(-1) &\to -c^{p_2}(a_s^{\pm} p_1 2^{-p_1} e^{-p_1 \pi i} + p_2 2^{-p_2} e^{-p_2 \pi i}) \end{aligned}\right\} \tag{4.19}$$

as $c \to 0$ and hence that their ratio approaches a finite limit. Thus, the limiting form of the characteristic equation is obtained in the form

$$\begin{aligned} \Delta_1(\alpha_s, z_s; Ri) &= \lim_{c\to 0}\{(1+y_c)\Delta(\alpha, c, z; Ri)\} \\ &= p_1 - p_2 + p_1 p_2\{F(z_s, p_1) - F(z_s, p_2)\} \\ &\qquad + A_s(a_s, Ri)\{p_1 F(z_s, p_1) - p_2 F(z_s, p_2)\} = 0, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} A_s(a_s, Ri) &= \lim_{c\to 0}\{(1+y_c)\Phi'(-1)/\Phi(-1)\} \\ &= -\frac{a_s p_1 e^{-(p_1-p_2)\pi i} + p_2 2^{p_1-p_2}}{a_s e^{-(p_1-p_2)\pi i} + 2^{p_1-p_2}} \end{aligned} \tag{4.21}$$

and the roots p_1 and p_2 are to be evaluated for $c = 0$, i.e. $\eta = Ri$. For $0 \leqslant Ri < 0{\cdot}0554$, equation (4.20) admits pairs of solutions $(a_s^{\pm}, z_s^{\pm})$ as given in table 2; for $Ri = 0{\cdot}0554$, $a_s^+ = a_s^-$ and $z_s^+ = z_s^-$; and, for $Ri > 0{\cdot}0554$, no real solutions exist and we have complete stability.

These results show that

$$R^{\frac{1}{3}} \sim (2^{\frac{2}{3}} z_s^{\pm} / \alpha_s^{\frac{1}{3}}) c^{-1}, \tag{4.22}$$

as $c \to 0$ along the curves of neutral stability. It is also not difficult to show that $c \to K_s^{\pm}(\alpha - \alpha_s)$ as $\alpha - \alpha_s \to 0$ (cf. figure 3) and hence that

$$R^{\frac{1}{3}} \sim (2^{\frac{2}{3}} z_s^{\pm} / \alpha_s^{\frac{1}{3}} K_s^{\pm})(\alpha - \alpha_s)^{-1} \quad \text{as} \quad \alpha - \alpha_s \to 0. \tag{4.23}$$

The constants $K_s^{\pm}$ could, in principle, be determined analytically by the methods described by Hughes & Reid (1968), but the analysis is too complicated to be of

Ri	α_s	a_s^-	z_s^-	a_s^+	z_s^+
0·00	0·000	4·592	2·297	2·000	∞
0·01	0·275	4·313	2·420	1·982	5·073
0·02	0·390	4·023	2·555	1·901	4·645
0·03	0·479	3·713	2·707	1·884	4·378
0·04	0·555	3·364	2·887	1·922	4·136
0·05	0·622	2·921	3·134	2·060	3·850
0·052	0·635	2·801	3·207	2·118	3·772
0·054	0·648	2·648	3·307	2·210	3·667
0·0554	0·656	2·41	3·48	2·41	3·48

TABLE 2. The values of the parameters associated with the limiting solution as $R \to \infty$ along the curves of neutral stability.

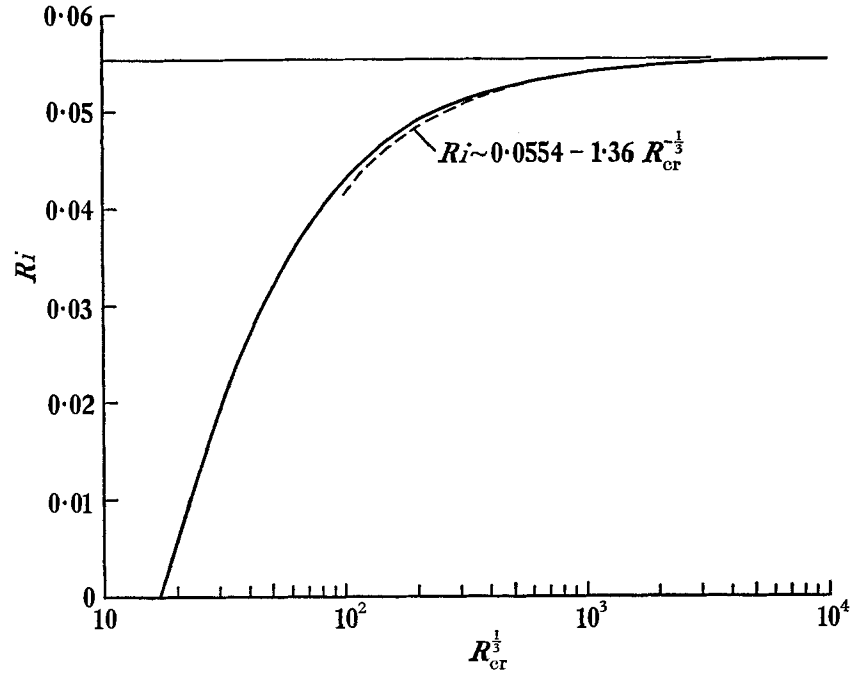

FIGURE 4. The variation of the minimum critical Reynolds number with the Richardson number.

any real interest. To obtain the asymptote to the curve shown in figure 4 as $R_{cr} \to \infty$ it is necessary first to obtain the dependence of Ri on c as $R_{cr} \to \infty$. No attempt was made to do this analytically but from the numerical results it was found that

$$Ri \sim 0{\cdot}0554 - 0{\cdot}214c \quad \text{as} \quad c \to 0 \tag{4.24}$$

and this result, together with equation (4·23), leads to the desired asymptote

$$Ri \sim 0{\cdot}0554 - 1{\cdot}36 R_{cr}^{-\frac{1}{3}} \quad \text{as} \quad R_{cr} \to \infty. \tag{4.25}$$

5. Concluding remarks

One of the most striking features of the present results is the sharp transition that was found between the thermal mode of instability which sets in at a value of the Rayleigh number that is independent of the shear and the Tollmien–Schlichting mode of instability which appears only at very high shear rates. This transition occurs at a small negative value of the Richardson number and it is tempting, therefore, to suggest that this phenomenon may be related to the transition that is observed in heat flux measurements in the atmospheric boundary layer (see, for example, Priestley 1959; or Townsend 1962). To obtain a quantitative comparison between theoretical and observational results, however, it will clearly be necessary to consider the stability of stratified shear flows for more general classes of temperature and velocity profiles.

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